Static- and dynamic-circuit models of PWM buck-derived DC–DC convertors

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Abstract: A systematic derivation of DC-circuit and small-signal-circuit models and characteristics for pulse-width-modulated buck-derived convertors operating in the continuous conduction mode is presented. Starting from voltage- and current-transfer functions of an idealised switching part of the convertor and using the linearisation technique, pure linear-circuit models of lossless convertors are derived. To obtain the models of lossy convertors, the principle of energy conservation is used to derive an equivalent averaged resistance of all parasitic resistances and an equivalent averaged voltage of offset voltage sources of switches. The advantage of the models is that they can be directly utilised in standard circuit simulators. The models are especially convenient in analysing complicated convertor topologies and for including parasitic components. The proposed models lead to the same characteristics as those obtained by means of the state-space-averaging method. As an example, a forward convertor is analysed, taking into account all parasitic resistances and threshold voltages of diodes. A complete set of small-signal characteristics is given for this convertor. Models and expressions for DC and small-signal characteristics that account for parasitic resistances are also given for other multiswitch and transformer buck-derived convertors such as push–pull, half-bridge, and full-bridge circuits.

1 Introduction

The state-space-averaging method introduced by Middlebrook and Cuk [1] is widely used to derive expressions for small-signal characteristics of pulse-width-modulated (PWM) convertors [1–7]. This general method has led to understanding of the dynamic performance of PWM convertors. However, the state-space method is sometimes tedious, especially when the convertor equivalent circuit contains a large number of elements. The purpose of this paper is to present a simple, systematic method of obtaining DC-circuit and small-signal-circuit models of PWM buck-derived convertors operating in the continuous-conduction mode (CCM). The starting point of derivation of the models is the voltage- and current-transfer functions of the switching part of the convertor which are perturbed and linearised about the steady-state operating point. The resulting relationships between voltages and currents are used to develop a linear-circuit model of the convertor-switching part comprised of dependent sources. To take into account parasitic resistors and offset-voltage sources of switches (which are of great importance in low-voltage high-current applications), the equivalent averaged resistance (EAR) of switched resistors and the equivalent averaged voltage (EAV) of switched voltage sources are determined using the principle of energy conservation. The inductances and capacitances are not averaged because the inductor currents and the capacitor voltages are continuous functions. The models can be used to find the convertor-performance static and dynamic characteristics either analytically or numerically. They offer three important advantages. First, the switching part of the convertor may consist of any number of switches, e.g. transistors and diodes. Secondly, the linear part of the convertor may consist of any number of elements, e.g. parasitic components and/or additional input and output filters. Third, circuit-simulation programs can be used for analysing the convertor models. Other interesting approaches to obtaining circuit-convertor models that are compatible with general-purpose CAD circuit-simulation packages such as SPICE have been presented recently by Griffin [8] and Kimhi and Ben-Yaakov [9].

2 Analysis of forward convertor

2.1 Assumptions

The analysis of the forward convertor of Fig. 1a is carried out under the following assumptions:

(i) The transistor output capacitance and diode capacitances are neglected; therefore, switching losses are assumed to be zero.
(ii) The transistor ON resistance $r_{ds}$ is linear and the transistor OFF resistance is infinite.
(iii) The diodes in the ON state are modelled by a linear battery $V_F$ and a linear forward resistance $R_F$ and in the OFF state by an infinite resistance.
(iv) The transformer-leakage inductances, the magnetising inductance, the stray capacitances and the magnetic-core parallel resistance are neglected.
(v) The diode $D_3$ is ideal.

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(vi) Passive components are linear, time-invariant and frequency-independent.

(vii) The inductor current is constant during the entire switching period.

\[
M_i \equiv \frac{I_k}{I_i} = \frac{n}{D}
\]

where \( n \) is the transformer turns ratio, \( V_i \) is the DC component of the input voltage equal to DC component of the ideal switch, and \( V_T \) is the DC component of the voltage across the diode \( D_2 \) if the diodes and the transformer are ideal. Assuming that all the parasitic resistances are zero and \( V_p = 0 \), \( M_p \) becomes the DC-voltage transfer function of the entire lossless converter. Since the switch \( S \) and the transformer in Fig. 1c are assumed to be ideal, the following is true: \( V_i I_T = V_T I_L \). Therefore, the DC-current transfer function is

\[
M_I \equiv \frac{I_k}{I_L} = \frac{n}{D}
\]

The output impedance of the input-voltage source is zero for both DC and AC components.

2.2 Circuit models of forward converter for DC and small-signal operation

A circuit of the forward converter is shown in Fig. 1a. It consists of a MOSFET as a controllable switch \( S \) (a BJT, an IGBT, or an MCT can also be used), an isolation transformer with a demagnetising winding, diodes \( D_1 \), \( D_2 \), and \( D_3 \), an inductor \( L \), a filter capacitor \( C \), and a load resistance \( R \). The switch is turned on and off at the switching frequency \( f_S = 1/T \) with the ON duty ratio \( D = t_{on}/T \), where \( t_{on} \) is the interval when the switch is ON. Fig. 1b depicts an equivalent circuit of the converter, where \( r_{DS} \) is the transistor ON resistance, \( r_T \) is the winding resistance of the primary of the transformer, \( r_{T2} \) is the winding resistance of the secondary of the transformer, \( V_T \) is the threshold voltage of the diode, \( R_p \) is the forward resistance of the diode \( D_2 \), and \( r_c \) is the ESR of the filter capacitor. It is assumed that the demagnetising winding of the transformer and the diode \( D_3 \) have no influence on a properly designed forward converter because of their small current. In the equivalent circuit of Fig. 1c, \( r_{DS} \) and \( r_T \) are reflected from the primary to the secondary side of the transformer.

Steady-state analysis of the circuit of Fig. 1c results in the DC-voltage transfer function:

\[
M_V \equiv \frac{V_L}{V_i} = \frac{D}{n}
\]

(viii) The output impedance of the input-voltage source is zero for both DC and AC components.

\[
v_T = (D/n + d/n)(V_T + v_i)
\]

Likewise, substituting eqns. 4, 7 and 6 into eqn. 10 one obtains

\[
I_T + i_T = \left( \frac{D}{n} + \frac{d}{n} \right) (I_L + i_i)
\]

Now an EAR of the switched components is introduced using the principle of energy conservation. The switch \( S \) and the diode \( D_1 \) are ON for the interval 0 < \( t < DT \). The switch current in the circuit of Fig. 1b is approximately constant and given by

\[
i_T \approx \frac{I_k}{n} \quad \text{for} \quad 0 < t < DT
\]

resulting in the RMS value of the switch current

\[
I_{rms} = \sqrt{\frac{1}{T} \int_{0}^{T} i_T^2 \, dt} = \sqrt{\frac{D}{n}} I_k
\]
The power dissipated in the resistors $r_{DS}$ and $r_{T1}$ on the primary side of the transformer

$$P_{ST1} = I_{Sm}^2 (r_{DS} + r_{T1}) = \frac{D(I_{DS}^2 + r_{T1})}{n^2}$$  \hspace{1cm} (15)$$
is equal to the power dissipated in the equivalent averaged resistance $r_{av}$ of $r_{DS}$ and $r_{T1}$ on the secondary side of the transformer. Hence

$$r_{av} = \frac{D(r_{DS} + r_{T1})}{n^2}$$  \hspace{1cm} (16)$$
Note that the output resistance of the input voltage source $V_i$ can be added in series with $r_{DS}$ and $r_{T1}$. Similar considerations reveal that

$$r_{T2av} = Dr_{T2}$$  \hspace{1cm} (17)$$
The EAR of the forward resistances $R_{F1}$ and $R_{F2}$ for the diodes $D_1$ and $D_2$ is obtained analogously:

$$R_{Fav} = DR_{F1} + (1 - D)R_{F2}$$  \hspace{1cm} (18)$$
If the forward resistances of the diodes are identical, the duty ratio $D$ does not affect their averaged value $R_{Fav}$ which is then equal to $R_F$. Thus, the power dissipated in the equivalent averaged resistances of the diodes and the secondary of the transformer is given by

$$P_{DT2} = I_{2av}^2 (R_F + Dr_{T2})$$  \hspace{1cm} (19)$$
The EAR of $r_{DS}$, $r_{T1}$, $r_{T2}$ and $R_F$ is obtained from eqns. 16–18:

$$r_{av} = r_{sav} + r_{T2av} + R_{Fav}$$

$$= \frac{D(r_{DS} + r_{T1})}{n^2} + R_F$$  \hspace{1cm} (20)$$
resulting in the total EAR in series with the inductor $L$.

$$r = r_{av} + r_L = \frac{D(r_{DS} + r_{T1})}{n^2} + r_{T2} + R_F + r_L$$  \hspace{1cm} (21)$$
The EAV of the forward batteries $V_{F1}$ and $V_{F2}$ of the diodes $D_1$ and $D_2$ is

$$V_{Fav} = D(V_{F1} + (1 - D)V_{F2})$$  \hspace{1cm} (22)$$
Assuming that the voltages of the batteries of the diodes are identical, $V_{Fav}$ is equal to $V_F$. Then the power dissipated in the equivalent averaged voltage source is

$$P_{VF} = V_F I_L$$  \hspace{1cm} (23)$$
Let us assume that the magnitudes of the AC components in 3–8 are much lower than the DC components. Hence, the products $(d/n)i_i$ and $(d/n)i_i$ in (eqns. 11 and 12) can be neglected. Using eqns. 16, 17, 18 and 21, a linear-circuit model of the convertor for DC and small-signal operation is obtained and shown in Fig. 2a. Since this model is linear, the principle of superposition can be used to derive the DC model shown in Fig. 2b and the small-signal model shown in Fig. 2c.

It is apparent from Fig. 2c that $t_0 = 0$ if $(V_i/n)d = -(D/n)i_i$. Accordingly, the control law is $d = -(D/V_i)i_i$.

From Fig. 2b, the DC-voltage transfer function of the lossy convertor is found to be expressed as

$$M_{VDC} \equiv \frac{V_o}{V_i} = \frac{D}{n} \left[ \frac{1}{n} + \frac{1}{1 + (r/R)} \right]$$

$$= \frac{D}{n} \left[ \frac{1}{1 + (r/R)} + (V_o/V_i) \right]$$  \hspace{1cm} (24)$$
Hence, one obtains the efficiency of the convertor

$$\eta = \frac{V_o I_o}{V_i I_i} = M_{VDC} \frac{I_i}{I_f} = \frac{nV_{DC}}{D}$$

$$= \left(1 - \frac{V_F}{D V_i} \right) \left[ \frac{1}{1 + (r/R)} + (V_o/V_i) \right]$$  \hspace{1cm} (25)$$
Note that the power loss in the ESR of the filter capacitor and switching losses are neglected. Eqns. 24 and 25 are illustrated in Figs. 3 and 4.

In the above derivations as well as in the state–space-averaging method, the inductor current is assumed to be constant, resulting in zero current through the filter capacitor and zero conduction loss in the ESR. In reality, the current through the filter capacitor is approximately equal to the AC component of the inductor current given by

$$i_c = \begin{cases} \frac{\Delta i_f}{DT} t - \frac{\Delta i_f}{2} & \text{ for } 0 < t \leq DT \\ -\frac{\Delta i_f}{(1-D)T} (t - DT) + \frac{\Delta i_f}{2} & \text{ for } DT < t \leq T \end{cases}$$  \hspace{1cm} (26)$$
where $\Delta i_f$ is the peak-to-peak ripple current of the inductor. The RMS value of the capacitor current is

$$I_{Crms} = \frac{\Delta i_c}{\sqrt{12}} \frac{V_o (1-D)}{\sqrt{12} f_i L}$$  \hspace{1cm} (27)$$
leading to the power-conduction loss in the filter capacitor

\[ P_{rc} = r_c I_{rms}^2 = \frac{r_c V_o^2 (1 - D)^2}{12 (f_s L)^2} \]  

(28)

The switching losses in the MOSFET can be approximated by

\[ P_{sw} = f_s C_s V_o^2 = \frac{f_s C_s R}{M_f V_{DC}} P_o \]  

(29)

where \( C_s \) is the transistor-output capacitance (assumed to be linear) and \( P_o \) is the DC output power. The power loss in the inductor is

\[ P_{rl} = r_L I_d^2 \]  

(30)

Using eqns. 15, 19, 23, 28, 29 and 30 gives the total power loss

\[ P_D = P_{ST1} + P_{DT2} + P_T + P_{rc} + P_{sw} + P_{rl} \]  

(31)

Hence, one arrives at the efficiency of the convertor taking also into account the power-conduction loss in the filter capacitor and the switching losses in the MOSFET neglected in eqn. 25:

\[ \eta = \frac{P_o}{P_o + P_D} \]

(32)

The models of Fig. 2 exhibit an important advantage. They consist of standard circuit elements and can therefore be analysed using circuit-simulation programs. Many previous models of the convertor contain a DC transformer [1, 2, 4, 5] which is not a circuit component and therefore cannot be handled by circuit-oriented simulators.

The above method of obtaining the linear models of the forward convertor is particularly convenient for more complex convertor topologies, e.g. for more sophisticated output filters and/or more complicated loads and/or other parasitic parameters (for instance, the equivalent series inductance of the capacitor).

2.3 Small-signal characteristics of forward convertor

The small-signal model of Fig. 2c can be used to describe the convertor performance for frequencies \( f \) up to about one-half the switching frequency \( f_s \). Referring to that model, the control-to-output (or duty ratio-to-output) transfer function in the s-domain is expressed as

\[ T_p(s) = \frac{v_o(s)}{d(s)} \]

(33)

where

\[ \omega_s = \frac{1}{C r C} \]  

(34)

\[ \xi_s = \frac{C (R_r + r_r r_s) + L}{2 \sqrt{LC(R + r_s)(R + r)}} \]  

(35)

\[ \omega = \sqrt{\frac{R + r}{LC(R + r_s)}} \]  

(36)
For $\xi > 1$, the frequencies of the real poles are
\[ f_{p1}, f_{p2} = f_s \left( \xi \pm \sqrt{\xi^2 - 1} \right) \] (37)
where $f_s = \omega_i / (2\pi)$.

The input-to-output (or line-to-output) voltage-transfer function, also called an open-loop dynamic line regulation or an audio susceptibility (which describes the input–output noise transmission), is
\[
M_a(s) = \frac{v_i(s)}{i_d(s)} = \frac{DRr_c}{nL(R + r_c)} \times \frac{s + (1/C_r)}{s^2 + s \frac{C(R_r + R + r_c) + L}{LC(R + r_c)} + R + r} = \frac{DRr_c^2}{nL \omega o_s \left( s + 2 \omega o \omega o_s + \omega o^2 \right)} \frac{D^2 \omega o^2 + \omega o_r}{s + \omega o_r} \] (38)
The open-loop input impedance is
\[
Z_{i}(s) = \frac{-v_i(s)}{i_d(s)} \bigg|_{s=0} = \frac{nL}{D^2} \frac{s^2 + \frac{C(R_r + R + r_c) + L}{LC(R + r_c)}} + \frac{R + r} {\frac{1}{C(r + r_c)}} = \frac{nL \omega o_s \left( s + 2 \omega o \omega o_s + \omega o^2 \right)} {D^2 \omega o^2 + \omega o_r} \] (39)
where
\[ \omega o_r = \frac{1}{C(R + r_c)} \] (40)
The open-loop output impedance of the unloaded converter (i.e. excluding the load resistance $R$) is
\[
Z_p(s) = \frac{-v_i(s)}{i_d(s)} \bigg|_{s=0} = \frac{DRr_c}{R + r} \times \frac{s^2 + \frac{C(r_c + r_c + L)}{LC(R + r_c)}} {r_c \frac{r_c + r} {\frac{1}{LC}}} = \frac{DRr_c^2 \left( s + \omega o \omega o_s + \omega o^2 \right)} {\left( R + r \right) \omega o_s \left( s + 2 \omega o \omega o_s + \omega o^2 \right)} \] (41)
where
\[ \omega o = \frac{r}{L} \] (42)
\[ \xi_o = \frac{r_c + r} {2 \sqrt{L/C}} \] (43)
\[ \omega o = \frac{1}{\sqrt{LC}} \] (44)
If $r = r_c = \sqrt{L/C}$, then $Z_p(s) = r$. In this case, $Z_p$ is independent of frequency.

The open-loop output impedance of the loaded converter (i.e., including the load resistance $R$) is
\[
Z_p(s) = \frac{-v_i(s)}{i_d(s)} \bigg|_{s=0} = \frac{DRr_c}{R + r} \times \frac{s^2 + \frac{C(r_c + r_c + L)}{LC(R + r_c)}} {r_c \frac{r_c + r} {LC}} = \frac{DRr_c^2 \left( s + \omega o \omega o_s + \omega o^2 \right)} {\left( R + r \right) \omega o_s \left( s + 2 \omega o \omega o_s + \omega o^2 \right)} \] (45)

The open-loop dynamic load regulation is defined as
\[
\delta_a(s) = \frac{-v_i(s)}{i_d(s)} \bigg|_{s=0} = \frac{nL \omega o_s \left( s + 2 \omega o \omega o_s + \omega o^2 \right)} {D^2 \omega o^2 + \omega o_r} \] (46)
Setting $s = 0$, one obtains the low-frequency asymptotes of the above quantities:
\[
T_a(0) = \frac{V_o}{V_i} \frac{R}{n} \frac{R + r} {R + r_c} \] (47)
\[
M_a(0) = \frac{D}{n} \frac{R}{R + r_c} \] (48)
\[
Z_a(0) = \frac{n(R + r)} {D^2} \] (49)
\[
Z_p(0) = r \] (50)
\[
Z_p(0) = \frac{Rr_c}{R + r_c} \] (51)

Note that all the above expressions are dependent upon $r$. Eqn. 48 is different from eqn. 24 because $V_o$ has been assumed to be a short circuit for a small-signal analysis.

As $s \to \infty$, one obtains the high-frequency asymptotes of the small-signal characteristics:
\[
T_a(\infty) = 0 \] (52)
\[
M_a(\infty) = 0 \] (53)
\[
Z_a(\infty) = \infty \] (54)
\[
Z_p(\infty) = r_c \] (55)
\[
Z_p(\infty) = \frac{Rr_c}{R + r_c} \] (56)

3 Plots of small-signal characteristics of forward converter

A forward converter of Fig. 1a was designed to meet the following specifications: $V_i$ = 40 to 60 V, $V_o$ = 5 V, $I_o$ = 2.5 to 20 A, $f_s$ = 200 kHz, and $V_o / V_i \leq 1\%$, where $V_i$ is the peak-to-peak output ripple voltage. A design procedure based on Reference 5 leads to the circuit parameters: $R = 0.25$ to 2 $\Omega$, $n = 2$, $D = 0.2$ to 0.5, $L = 5 \mu\text{H}$, $C = 1 \text{mF}$, $r_c = 50 \text{m}\Omega$, $r = 40 \text{m}\Omega$, $r_{DS} = 0.5 \Omega$, $r_T1 = 50 \Omega$, $r_T2 = 25 \Omega$, $R_p = 25 \text{m}\Omega$, and $V_o = 0.7 \text{V}$. The model parameter $r$ calculated from eqn. 21 for $D = 0.5$ is 0.15 $\Omega$. The aim of this Section is to show the influence of parasitic resistances $r$ and $r_c$, and the load resistance $R$ on control-to-output transfer function $T_p$, input-to-output voltage-transfer function $M_p$, open-loop input impedance $Z_i$ and open-loop output impedance of the
Although the characteristics are valid up to one-half of the switching frequency, the plots are drawn to 1 MHz so that they may be useful when the poles and zeros are shifted to lower frequencies. The following circuit parameters are held constant in all calculations: \( L = 5 \mu \text{H}, \ C = 1 \text{ mF} \) and \( n = 2 \). Plots were calculated for fixed values of two of these resistances and various values of the third one. The fixed values were: \( R = 0.25 \Omega, \ r = 0.15 \Omega \) and \( r_c = 0.05 \Omega \). The minimum value of \( R \) results in the highest sensitivity of the characteristics to variations in \( r \) and \( r_c \). The value of \( r_c \) given above is quite typical for electrolytic capacitors. The ranges used were: from 0.25 \( \Omega \) to 2 \( \Omega \) for \( R \), from 0.05 \( \Omega \) to 0.2 \( \Omega \) for \( r \) and from 0.01 \( \Omega \) to 0.1 \( \Omega \) for \( r_c \).

Fig. 5 shows the magnitude and the phase of the control-to-output transfer function \( T_p \) against frequency for \( V_f = 60 \text{ V}, \ n = 2, L = 5 \mu \text{H}, \ C = 1 \text{ mF}, \ R = 0.25 \Omega, \ r_c = 0.05 \Omega \). The maximum input voltage \( V_f = 60 \text{ V} \) was selected because it results in the highest crossover frequency. As \( r_c \) was increased, the crossover frequency increased from about 10 to 70 kHz and the minimum phase was below \(-90^\circ\) only for small values of \( r_c \). For higher values of \( r_c \), the circuit behaved like a 1st-order system. The values of the damping factor were \( \xi_r \) = 1.02, 1.15, 1.28 and 1.56. The corner frequencies were \( f_c \) = 2.79, 2.69, 2.60, and 2.41 kHz. The frequencies of the poles were \( f_p \) = 2.29, 1.56, 1.26, and 0.87 kHz, and \( f_p \) = 3.40, 4.63, 5.37, and 6.63 kHz, and the frequencies of the zero were \( f_z \) = 15.92, 5.30, 3.18, and 1.59 kHz, respectively.

Fig. 6 depicts plots of \( T_p \) for \( V_f = 60 \text{ V}, \ R = 0.25 \Omega, \ r = 0.15 \Omega \) and \( r_c = 0.01, 0.03, 0.05 \) and 0.1 \( \Omega \). The maximum input voltage \( V_f = 60 \text{ V} \) was selected because it results in the highest crossover frequency. As \( r \) was increased, the crossover frequency increased from about 10 to 70 kHz and the minimum phase was below \(-90^\circ\) only for small values of \( r \). For higher values of \( r \), the circuit behaved like a 1st-order system. The values of the damping factor were \( \xi_r \) = 0.77, 1.04, 1.28 and 1.49. The corner frequencies were \( f_c \) = 2.25, 2.43, 2.60 and 2.76 kHz. The frequencies of the poles in the case of \( \xi_r \geq 1 \) were \( f_{p\xi} \) = 1.86, 1.27 and 1.06 kHz, and \( f_{p\xi} \) = 3.18, 5.37 and 7.16 kHz, and the frequency of the zero was \( f_z \) = 3.18 kHz.

Characteristics of \( T_p \) are plotted in Fig. 7 for \( V_f = 60 \text{ V}, n = 2, L = 5 \mu \text{H}, \ C = 1 \text{ mF}, \ R = 0.25, 0.5 \) and 2 \( \Omega \). The effect of \( R \) on \( T_p \) was almost negligible. The values of
the damping factor were $\xi = 1.28, 1.33$ and 1.39. The corner frequencies were $f_1 = 2.60, 2.45$ and 2.31 kHz. The frequencies of the poles were $f_{p1} = 1.26, 1.11$ and 0.98 kHz, and $f_{p2} = 5.37, 5.40$ and 5.43 kHz, and the frequency of the zeros was $f_0 = 3.18$ kHz, equal to that for the case shown in Fig. 6.

The phase characteristics and the shape of the magnitude characteristics of the input-to-output voltage-transfer function $M_\nu$ are the same as for $T_\nu$. To obtain the exact magnitude characteristic of $M_\nu$ from the magnitude characteristic of $T_\nu$, one should shift it down by a value of $20 \log (V_\nu/D)$. An example plot of the magnitude of the input-to-output transfer function $M_\nu$ against frequency for $V_\nu = 60$ V, $D = 0.35$, $n = 0.25 \Omega$, $r = 0.15 \Omega$ and $r_c = 0.01, 0.03, 0.05$ and 0.1 $\Omega$ is given in Fig. 8.

Figs. 9–11 show plots of the open-loop input impedance $Z_i$ against frequency. As shown, $|Z_i|$ was almost independent of $r_c$, slightly increased with $r$ at low frequencies and appreciably increased with $R$ at low frequencies. The frequencies of the poles for the case shown in Fig. 9 were $f_{p1} = 0.61, 0.57, 0.53$ and 0.45 kHz, and the frequencies of the zeros were the same as the frequencies of the poles in the case shown in Fig. 5.

Characteristics of the open-loop output impedance of the loaded converter are depicted in Figs. 12–14. $|Z_o|$ increased with increasing $r$ and $R$ at low frequencies and

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<th>$Z_o$</th>
<th>$f$</th>
<th>$\Phi_{Z_o}$</th>
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<td>100</td>
<td>-90</td>
<td>0</td>
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Fig. 8  Amplitude of input-to-output transfer function $|M_\nu|$ against $f$

$V_\nu = 60$ V, $n = 2$, $L = 5 \mu$H, $C = 1$ mF, $R = 0.25 \Omega$, $r = 0.15 \Omega$

- $r_c = 0.01 \Omega$
- $r_c = 0.03 \Omega$
- $r_c = 0.05 \Omega$
- $r_c = 0.1 \Omega$

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<th>$IZ_i$</th>
<th>$f$</th>
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<td>100</td>
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Fig. 10  Open-loop input impedance $Z_i = |Z_i| \exp (j\Phi_{Z_i})$

$D = 0.35$, $n = 2$, $L = 5 \mu$H, $C = 1$ mF, $r = 0.15 \Omega$

- $r_c = 0.05 \Omega$
- $r_c = 0.1 \Omega$
- $r_c = 0.15 \Omega$
- $r_c = 0.2 \Omega$

| $|Z_o|$ | $f$ |
|-------|----|
| 100   | -90|

Fig. 11  Open-loop input impedance $Z_o = |Z_o| \exp (j\Phi_{Z_o})$

$D = 0.35$, $n = 2$, $L = 5 \mu$H, $C = 1$ mF, $r_c = 0.05 \Omega$, $r = 0.15 \Omega$

- $R = 0.25 \Omega$
- $R = 0.5 \Omega$
- $R = 2.0 \Omega$

| $|Z_o|$ | $f$ |
|-------|----|
| 100   | -90|
increased with increasing $r_c$ at high frequencies. The frequencies of zeros $f_i$ and the frequencies of the pole $f_p$ were the same as those shown in Figs. 5-7, respectively. The frequencies of the zero $f_i$ were: 4.77 kHz for the case shown in Fig. 13.

The ESL affects the small-signal characteristics in the high-frequency range, especially the control-to-output transfer function and the open-loop output impedance of the loaded convertor. Figs. 15 and 16 show magnitudes and phases of $T_p$ and $Z_o$ against frequency for $V_i = 60$ V, $D = 0.35$, $R = 0.25$ $\Omega$, $r_c = 0.05$ $\Omega$, $r = 0.15$ $\Omega$, and $L_c = 1, 10, 50$ and 100 nH.

4 Effect of ESL of filter capacitor on small-signal characteristics

The equivalent circuit of the capacitor consists of a series combination of its capacitance $C$, the ESR $r_c$, and the equivalent series inductance (ESL) $L_c$. Using this equivalent circuit of the filter capacitor in the model of Fig. 2c, small-signal characteristics of the forward convertor have been derived and are

$$ T_p(s) \left|_{s=0} \right. = \frac{v_i(s)}{v(s)} = \frac{V_i R n L}{s^3 + s^2 + (R + r_c) L + (R + r_c) L \frac{s^2 + s r_c / L_c + 1 / (L C)}{L L_c} + s} \frac{R C + R + r_c r + (L / C)}{L C} + R + r $$

$$ M_o(s) \left|_{s=0} \right. = \frac{v_i(s)}{i(s)} = \frac{D R n L}{s^3 + s^2 + (R + r) L + (R + r) L \frac{s^2 + s r_c / L_c + 1 / (L C)}{L L_c} + s} \frac{R C + R + r_c r + (L / C)}{L L_c} + R + r $$

$$ Z_o(s) \left|_{s=0} \right. = \frac{v_i(s)}{i(s)} = \frac{n L}{D^2} \frac{s^2 + s r C + r C L + \frac{1}{L C} + r}{s^2 + s^2 (R + r) L + (R + r) L \frac{s^2 + s r_c / L_c + 1 / (L C)}{L L_c} + s} \frac{R C + R + r_c r + (L / C)}{L L_c} + R + r $$

5 Models of other buck-derived convertors

5.1 Buck convertor

A model for the DC and small-signal operation of the buck convertor shown in Fig. 17 can be derived from the model of the forward convertor of Fig. 2 by setting $n = 1$, $r_{T1} = r_{T2} = R_{F1} = 0$, and $R_{F2} = R_p$ and replacing $V_F$ by

$$ V_i(s) \left|_{s=0} \right. = \frac{v_i(s)}{v(s)} = \frac{V_i R}{s^3 + s^2 + (R + r_c) L + (R + r_c) L \frac{s^2 + s r_c / L_c + 1 / (L C)}{L L_c} + s} \frac{R C + R + r_c r + (L / C)}{L C} + R + r $$
\( (1 - D) V_F \). Hence, eqns. 16, 18, 20 and 21 become \[ 10 \] respectively,

\begin{align*}
    r_{sw} &= D r_{DS} \\
    R_{Fav} &= (1 - D) R_{F2} = (1 - D) R_F \\
    r_s &= r_{sw} + R_{Fav} = D r_{DS} + (1 - D) R_F \\
    r &= r_s + r_L = D r_{DS} + (1 - D) R_F + r_L
\end{align*}

The small-signal characteristics given by eqns. 33–46 remain the same. The only difference is the value of \( r \).

The DC-voltage transfer function and the efficiency are

\[ M_{v,DC} = \frac{V_o}{V_i} = \left\{ \frac{D - V_F(1 - D)}{V_i} \right\} \frac{R}{R + r} \]

\[ \eta = \frac{V_o V_o}{V_i I_i} = \left\{ \frac{1 - (1 - D)V_F}{D V_i} \right\} \frac{R}{R + r} \]

\[ \eta = \left\{ \frac{1}{1 + (r/R) + (1 - D)(V_F/V_o)} \right\} \]

5.2 Push–pull convertor

For the push–pull convertor depicted in Fig. 18, the models from Fig. 2 are still valid \[ 11 \]. Assuming identical switches, identical diodes and a symmetrical transformer, one should only substitute \( 2D \) instead of \( D \) and calculate \( r \) from

\[ r = 2D \frac{r_{DS} + r_{T1}}{n^2} + \left( \frac{1}{2} + D \right) R_F + r_L \]
The DC-voltage transfer function and the efficiency are

\[ M_{\text{DC}} = \frac{V_o}{V_i} = \frac{2D}{n} \frac{V_o}{V_i} \frac{R}{R + r} \]

\[ \eta = \frac{V_o I_o}{V_i I_i} = \left( 1 - n \frac{V_o}{2DV_i} \right) \frac{R}{R + r} \]

\[ = \frac{1}{1 + (r/R) + (V_o/V_o)} \] (68)

5.3 Half-bridge convertor

The models for the half-bridge convertor shown in Fig. 19 are identical to those for the forward convertor.

Fig. 18 Push-pull convertor

Fig. 19 Half-bridge convertor

Fig. 20 Full-bridge convertor

However, \( r \) is the same as for the push-pull convertor and is given by eqn. 67. The DC-voltage transfer function and the efficiency are given by eqns. 24 and 25, respectively.

5.4 Full-bridge convertor

The full-bridge convertor depicted in Fig. 20 has the same models as the forward convertor. The EAR in series with the inductor is found to be

\[ r = 2D \frac{2D_{ns} + r_{T1}}{n^2} + \left( \frac{1}{2} + D \right) (R_P + r_{T2}) + r_L \] (69)

\[ = 20 \] (70)

Also, \( D \) must be replaced by \( 2D \) in expressions for DC and small-signal characteristics. The DC-voltage transfer function and the efficiency are the same as for the push-pull convertor and are given by eqns. 68 and 69, respectively.

6 Conclusions

The principle of energy conservation has been used to derive DC and small-signal linear models of PWM buck-derived convertors operating in CCM. As an example of the application of the models, a detailed analysis of the forward convertor has been presented. The obtained small-signal characteristics are the same as those derived using the state-space averaging method. Experimental results confirming these characteristics can be found in Reference 1 and are not duplicated in this paper. The results have been extended to other multwitch and transformer buck-derived convertors such as push-pull, half-bridge, and full-bridge. All buck-derived convertors have the same models, but the expressions for the equivalent average resistance \( r \) are somewhat different for different convertors. The plots of open-loop small-signal characteristics are useful in designing controllers. It is shown that the ESR of the filter capacitor makes the control-to-output transfer function of buck-derived convertors similar to that of a 1st-order system. An important advantage of the new models is that they can be used in standard circuit-simulation programs such as SPICE because they do not contain noncircuit components (e.g. DC transformers). The convertors can be operated in CCM for any load if diodes are replaced with controllable bidirectional switches such as MOSFETs, which results also in improved dynamic performance because of a smaller inductor and a bidirectional power flow. In this case, the proposed models are still valid with appropriate modifications in expressions for EAR and EAV. The presented method may be used for modelling and analysing other DC-DC convertors and DC-AC invertors.

7 References

1 MIDDLEBROOK, R.D., and CUK, S.: 'Advances in switched-mode power conversion' (TSLaco, Pasadena, CA, 1981), vols 1 and 2
8 GRIFFIN, R.E.: 'Unified converter models for continuous and discontinuous conduction mode', IEEE Power Electronics Specialists Conference Record, Milwaukee, WI, USA, 26-29 June 1989, pp. 853-860
Consider the buck or forward converter. The current through the capacitor is approximately equal to the AC component of the inductor current. For the interval \(0 < t < DT\) when the switch is ON and the diode is OFF, the capacitor current is

\[i_C(t) = \frac{\Delta I_L t}{DT} - \frac{\Delta I_L}{2}\]  

resulting in the AC component of the voltage across the ESR

\[v_{ESR}(t) = r_C i_C = r_C \left(\frac{\Delta I_L t}{DT} - \frac{\Delta I_L}{2}\right)\]  

and the AC component of the voltage across the filter capacitance

\[v_f(t) = \frac{1}{C} \int_0^t i_C dt + v_c(0) = \frac{\Delta I_L}{2C} \left(\frac{t^2}{DT} - t\right) + v_c(0)\]  

where \(\Delta I_L\) is the peak-to-peak ripple current of the inductor. Hence, the sum of the two voltages is

\[v(t) = v_{ESR} + v_f = \frac{\Delta I_L t^2}{2CDT} + \Delta I_L \left(\frac{r_C}{DT} - \frac{1}{2C}\right) t - r_C \frac{\Delta I_L}{2} + v_c(0)\]  

The peak-to-peak value of the total voltage \(v\), is equal to the peak-to-peak value of the ESR voltage \(v_{ESR}\) if the voltage drop across the capacitance is sufficiently low. The derivative of the voltage \(v\), is

\[\frac{dv}{dt} = \frac{\Delta I_L t}{CDT} + \Delta I_L \left(\frac{r_C}{DT} - \frac{1}{2C}\right) - r_C \frac{\Delta I_L}{2}\]  

from which the minimum value of \(v\), occurs at

\[t_{min} = \frac{DT}{2} - r_C C\]  

The waveforms for the interval \(DT < t < T\) can be obtained in a similar manner. The maximum value of \(v\), occurs at

\[t_{max} = \frac{(1 + D)T}{2} - r_C C\]  

For steady-state operation the average value of \(v_C(t)\) is zero. Hence

\[v_C(0) = (2D - 1) \frac{\Delta I_L}{12f_C}\]  

The peak-to-peak ripple voltage is independent of the voltage across the filter capacitor and is determined only by the ripple voltage of the ESR if \(t_{min} \leq 0\) and \(t_{max} \leq DT\). This happens when

\[C \geq C_{min} \left(\frac{1 - D_{min}}{2r_C f_s}, \frac{D_{max}}{2r_C f_s}\right)\]  

If the condition described by eqn. 79 is satisfied, the peak-to-peak ripple voltage of the buck and forward converters is

\[V_r = r_C \Delta I_{L,max} = \frac{r_C V_d(1 - D_{min})}{f_s L}\]  

For push-pull, half-bridge, and full-bridge converters,

\[C \geq C_{min} = \max \left(\frac{0.5 - D_{min}}{2r_p f_s}, \frac{D_{max}}{2r_p f_s}\right)\]  

where \(D_{max} \leq 0.5\). If the condition described by eqn. 81 is met, the peak-to-peak ripple voltage \(V_r\) of these converters is given by

\[V_r = r_C \Delta I_{L,max} = \frac{r_C V_d(0.5 - D_{min})}{f_s L}\]  

Waveforms of \(v_{ESR}\), \(v_C\) and \(v\), are depicted in Fig. 21 for three values of the filter capacitances. In Fig. 21a, the peak-to-peak value of \(v\), is higher than the peak-to-peak value of \(v_{ESR}\) because \(C < C_{min}\). Fig. 21b and c show the waveforms for \(C = C_{min}\) and \(C > C_{min}\), respectively. For both these cases, the peak-to-peak voltages of \(v\), and \(v_{ESR}\) are equal to each other.

Note that the minimum value of inductance \(L\) is determined either by the boundary between the continuous and discontinuous conduction modes or by the ripple voltage.